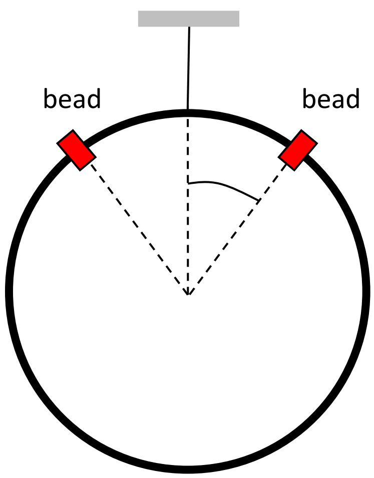
Teacher notes
Topic A

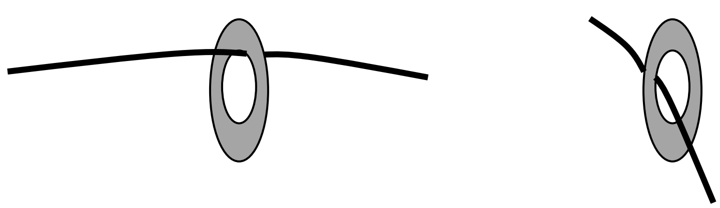
A nice circular motion problem with a nice math connection (And answering a comment by Chiara T.)

I firmly believe in making connections between different subjects whenever possible, especially between Physics and Math. This reinforces what one learns in one subject by putting things in context. This problem is one example of this.

A ring of mass M and radius R hangs from a string. Two beads, each of mass m start from rest at the same time from the top of the ring and slide down the ring.



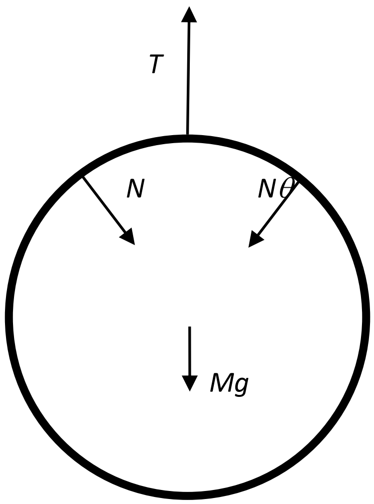
(a) Make a free body diagram for the ring and for one of the beads at the instant shown above.
(b) Show that the normal force on one of the beads is given by $N = mg(3\cos\theta - 2)$.
(c) Write down the condition that the tension in the string becomes zero.
(d) For what values of $\frac{m}{M}$ is the condition realized?
(e) For the least value of $\frac{m}{M}$ determine the angle at which the tension becomes zero.
(f) Repeat the analysis by considering an expression for the tension. You may refer to the fact that the bead may be in contact with the ring in more than one way as shown in the figure:

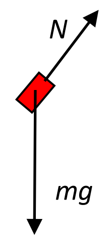


Find the net upward force on the ring when the tension becomes zero and comment on the result.

Answers

(a)





This assumes that the bead is in contact according to the first diagram in (f).

(b) For the ring: $T = Mg + 2N\cos\theta$

For one bead: $mg\cos\theta - N = \dfrac{mv^2}{R}$

Conservation of energy gives: $mgR(1-\cos\theta) = \dfrac{1}{2}mv^2 \Rightarrow v^2 = 2gR(1-\cos\theta)$.

Hence, for the bead equation,

$mg\cos\theta - N = \dfrac{m2gR(1-\cos\theta)}{R}$, giving $N = mg(3\cos\theta - 2)$.

(c) From the ring equation $T = 0$ implies $0 = Mg + 2N\cos\theta$. Substituting the value of the normal force we get $0 = Mg + 2mg(3\cos\theta - 2)\cos\theta$. This can be rewritten as

$$0 = 1 + 2\frac{m}{M}(3\cos\theta - 2)\cos\theta$$

(d) The condition above is $6\dfrac{m}{M}\cos^2\theta - 4\dfrac{m}{M}\cos\theta + 1 = 0$. It is realized, i.e. it has solutions, if the discriminant is non-negative i.e. $16(\dfrac{m}{M})^2 - 24\dfrac{m}{M} \geq 0$ or $(\dfrac{m}{M})(\dfrac{2m}{M} - 3) \geq 0$ which implies $m \geq \dfrac{3M}{2}$.

(e) The least value of $\frac{m}{M}$ is $\frac{3}{2}$ and the quadratic equation becomes $9\cos^2\theta - 6\cos\theta + 1 = 0$. The solution is $\cos\theta = \frac{1}{3}$ or $\theta \approx 71^\circ$.

(f) The force N does not always point downward on the ring as shown in (a). The force is given by $N = mg(3\cos\theta - 2)$. Initially $\theta = 0 \Rightarrow N = mg(3\cos 0 - 2) = mg$. For $\frac{m}{M} = \frac{3}{2}$, when $\cos\theta = \frac{1}{3} \Rightarrow N = mg(3\times\frac{1}{3} - 2) = -mg$ i.e. it reverses direction. The direction switch happens when $\cos\theta = \frac{2}{3} \Rightarrow \theta \approx 48^\circ$. The analysis above holds for $\theta < 48^\circ$ but also for $\theta > 48^\circ$ if we use absolute values. If you don't like absolute values, the analysis for $\theta > 48^\circ$ is:

For the ring: $T + 2N\cos\theta = Mg$

For one bead: $mg\cos\theta + N = \frac{mv^2}{R}$

Conservation of energy gives: $mgR(1-\cos\theta) = \frac{1}{2}mv^2 \Rightarrow v^2 = 2gR(1-\cos\theta)$.

Hence, for the bead equation,

$mg\cos\theta + N = \frac{m2gR(1-\cos\theta)}{R}$, giving $N = mg(2 - 3\cos\theta)$. Then

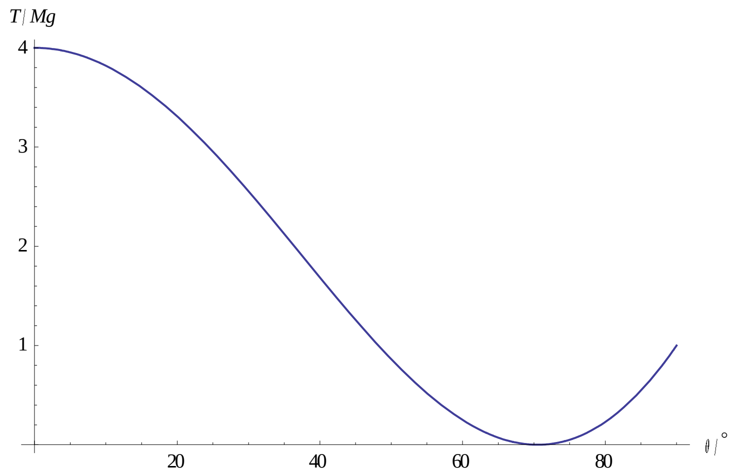
$$\begin{aligned} T &= Mg - 2N\cos\theta \\ &= Mg - 2mg(2-3\cos\theta)\cos\theta \end{aligned}$$

For, $\frac{m}{M} = \frac{3}{2}$, $T = Mg - 2\frac{3M}{2}g(2-3\cos\theta)\cos\theta = Mg\left(1 - 3(2-3\cos\theta)\cos\theta\right)$

The tension becomes zero when

$$\begin{aligned} 1 - 3(2-3\cos\theta)\cos\theta &= 0 \\ 9\cos^2\theta - 6\cos\theta + 1 &= 0 \\ \cos\theta &= \frac{1}{3} \end{aligned}$$

just as before. The graph shows the variation of the tension with angle showing that it becomes zero at $\theta \approx 71^\circ$. This graph assumes that the ring is always in equilibrium. But see below for what happens for $\theta > 71^\circ$.



Notice that the upward force on the ring is

$2N\cos\theta = 2mg(2-3\cos\theta)\cos\theta = 2mg\cos\theta - 6mg\cos^2\theta$. When is this a maximum?

$$\begin{aligned}\frac{d}{d\theta}2N\cos\theta &= \frac{d}{d\theta}(2mg\cos\theta - 6mg\cos^2\theta)\\ &= -2mg\sin\theta + 6mg\cos\theta\sin\theta\\ &= 0\end{aligned}$$

Solving we find $\sin\theta = 3\cos\theta\sin\theta \Rightarrow \cos\theta = \frac{1}{3}$. This shows that when the tension becomes zero the upward force is a maximum. The upward force on the ring in this case is

$$2N\cos\theta = 2mg(2-3\cos\theta)\cos\theta = 2mg(2-3\times\frac{1}{3})\times\frac{1}{3} = \frac{2mg}{3}.$$

So, if $\frac{m}{M} > \frac{3}{2}$, $2N\cos\theta > \frac{2\frac{3M}{2}g}{3}$, i.e. $2N\cos\theta > Mg$. The net force on the ring is then upwards and the ring will rise.